Dynamics semi-conjugated to a subshift for some polynomial mappings in $\mathbb{C}^2$

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2nd February 2008

Abstract

We study the dynamics near infinity of polynomial mappings f in $\mathbb{C}^2$. We assume that f has indeterminacy points and is non constant on the line at infinity L_∞. If L_∞ is f-attracting, we decompose the Green current along itineraries defined by the indeterminacy points and their preimages. The symbolic dynamics that arises is a subshift on an infinite alphabet.

MSC: 37F10, 37B10
Keywords: Polynomial mapping, Green current, subshift.

1 Introduction

We are interested in the dynamics of polynomial mappings f in $\mathbb{C}^2$ whose meromorphic extensions to $\mathbb{P}^2$ admit indeterminacy points and for which the line at infinity (which we denote by L_∞) is f-attracting (that is: there exists $C > 1$ such that for $p \in \mathbb{C}^2$ with $\|p\|$ large enough, one has $\|f(p)\| \geq C\|p\|$). In particular, given any large ball $\mathbb{B}$ in $\mathbb{C}^2$, these maps are polynomial-like in the sense of [DS03] from $f^{-1}(\mathbb{B})$ to $\mathbb{B}$. The dynamics is studied there: there exists an invariant probability measure which is K-mixing and of maximal entropy. Our goal is to study the dynamics near infinity, especially the structure of the *Green current*, which is a positive closed current of bidegree (1,1) invariant under the action of f^*.

In [DDS05], the authors consider the case where f_∞, the restriction of f to L_∞, is constant and they decompose the Green current into pieces associated to an itinerary defined by indeterminacy points. On the basin of attraction of the indeterminacy set, the itinerary map semi-conjugates f to a shift.

Another case which has been studied is when f admits a holomorphic extension to $\mathbb{P}^2$: in [BJ00], the authors showed the Green current admits a local laminar decomposition consisting of local stable manifolds of f to the Julia set of f_∞. Applying one dimensional theory, one also obtains in this case a dynamics semi-conjugated to a shift.

We study here a mixed situation. We assume that f admits indeterminacy points on L_∞ and that f_∞ is not a constant function. In order to describe clearly the new phenomena happening here, we consider the case where f_∞ is hyperbolic. The method we use allows to study more general cases. We will complete our study by giving several examples. In the hyperbolic case, we show that the Green current decomposes along some itineraries defined by the indeterminacy points and theirs preimages. Surprisingly, the local stable manifolds associated to the Julia set of f_∞ are not charged by the Green current. Furthermore, the symbolic dynamics we obtain is a subshift (a Markov chain), which is new for polynomial mappings.

The main tools we use are horizontal-like maps and a theorem of convergence of currents proved in [Duj04] and [DDS05]. Roughly speaking, such applications are contracting in the vertical direction and expanding in the horizontal one in a geometrical sense. For the reader's convenience, we give the basic properties of those objects.

Next, we define and study the basic properties of the family $\mathcal{G}$ of maps we consider. We give a simple sufficient condition for a map f to be in $\mathcal{G}$ and we prove the algebraic stability. Then, by a theorem of Sibony [Sib99], one can associate to f a natural invariant current (Green current). We give an easily computable formula for the trace of the Green current at infinity. This trace is a probability measure which is a combination of Dirac masses at the indeterminacy points and theirs preimages. Under some additional hypothesis, we also compute the topological degree.

We then study the decomposition of the Green current on a neighborhood of infinity under the hypotheses that the indeterminacy set is located in the Fatou set of f_∞, with no indeterminacy point being periodic for f_∞ and that f_∞ is hyperbolic. This set of maps contains an open subset of $\mathcal{G}$. The decomposition of the Green current semi-conjugates f to a subshift on an infinite alphabet. Under some additional hypothesis, we show that the range of the *escape rate* (which measures the asymptotic speed at which a point goes to infinity) is a full interval which is new for polynomial maps and we compute a mean escape rate. We will explain briefly how to obtain a weaker decomposition of the Green current in a more general case. Finally, we study examples, in particular the case where the indeterminacy points are located in the exceptionnal set of f_∞, in this case the support of the Green current is strictly contained in the Julia set of f.

2 Polynomial maps with dynamics at infinity

2.1 Horizontal-like maps

We recall here the facts we use on horizontal-like maps. Proofs and details can be found in [Duj04] and [DDS05].

Let $\mathbb{D}$ (resp. $\mathbb{D}_r$) be the unit disk (resp. the disk of radius r centered at 0) in

$\mathbb{C}$. Let Δ be the unit bidisk in $\mathbb{C}^2$, we denote its vertical boundary by $\partial_v\Delta$, and its horizontal boundary by $\partial_h\Delta$. Namely:

$\partial_v\Delta = \{(z,w) \in \mathbb{C}^2,\ |z| = 1,\ |w| < 1\}$ and $\partial_h\Delta = \{(z,w) \in \mathbb{C}^2,\ |z| < 1,\ |w| = 1\}$.

We have the following definitions:

Definition 2.1 *Let $\Delta_i \subset M_i$ be an open subset biholomorphic to Δ in the complex surface M_i for $i = 1, 2$. Let f be a dominating meromorphic map defined in some neighborhood of Δ_1 with values in M_2. The triple (f, Δ_1, Δ_2) defines a horizontal-like map if:*

- *f has no indeterminacy points in $\partial_v\Delta_1$ and $f(\partial_v\Delta_1) \cap \overline{\Delta_2} = \emptyset$;*
- *$f(\overline{\Delta_1}) \cap \partial\Delta_2 \subset \partial_v\Delta_2$;*
- *$f(\Delta_1) \cap \Delta_2 \neq \emptyset$.*

Definition 2.2 *A positive closed (1,1)-current T in Δ is* vertical *if:*

$$\mathrm{SuppT} \subset \mathbb{D}_{1-\varepsilon} \times \mathbb{D} \text{ for some } \varepsilon > 0.$$

Similarly, we can define horizontal currents.

We can define the (horizontal) slice measures m^{w_0} of a vertical positive closed (1,1)-current T by $T \wedge [w = w_0]$. These measures have the same mass, which we call the *slice mass* of T. The current T is zero if and only if its slice mass is zero. The main fact is that we can define the pull-back of such a current by a horizontal-like map, and we have the following:

Let (f, Δ_1, Δ_2) be a horizontal-like map then there exists a positive integer $d \geq 1$ such that for every vertical positive closed current T in Δ_2 of slice mass 1, $\frac{1}{d} f^*(T)$ is a vertical positive closed current in Δ_1 of slice mass 1.

We call this integer the degree of f, it can be computed as the intersection multiplicity of the preimage of a vertical line with a horizontal line. The following result is our main tool to obtain the convergence in Theorem 3.5:

Theorem 2.3 ([DDS05]) *Let $\{(f_i, \Delta_i, \Delta_{i+1})\}_{i\geq 1}$ be a sequence of horizontal-like maps of degree d_i such that $(f_i)^{-1}(\Delta_{i+1}) \subset \mathbb{D}_{1-\varepsilon} \times \mathbb{D} \subset \Delta_i$ for a fixed $\varepsilon > 0$. Assume that $K = \bigcap_{n\geq 1} f_1^{-1} \dots f_n^{-1}(\Delta_{n+1})$ has zero Lebesgues measure. For each n, let T_n be a vertical positive closed (1,1)-current of slice mass 1 in Δ_n.*

Then, the sequence of iterated pull-back $(\frac{1}{d_1} f_1^* \dots \frac{1}{d_n} f_n^* T_{n+1})_n$ converges to a vertical positive closed current τ of slice mass 1 in Δ_1 which is independent of (T_n).

2.2 The class $\mathcal{G}$

We are interested in the study of the polynomial mappings f of $\mathbb{C}^2$ for which L_∞ is attractive. Namely, there are constants $C > 1$ and M large enough such that for $\|p\| \geq M$, we have $\|f(p)\| \geq C\|p\|$. We also assume that the meromorphic extension of f to $\mathbb{P}^2$ admits indeterminacy points, we still denote by f that extension. The case where the restriction f_∞ of f to L_∞ is constant was studied in [DDS05], so we will assume that f_∞ is not constant. We denote by $\mathcal{G}$ the set of mappings satisfying the above properties (in fact, we are particularly interested in the case where f_∞ is hyperbolic and where the indeterminacy points are not periodic for f_∞).

Let $f = (f_1, f_2)$ of algebraic degree D be in $\mathcal{G}$. We denote by f_1^+ and f_2^+ the homogeneous parts of maximal degree. After a linear change of coordinates, we can assume $\deg f_1^+ = D$ and $\deg f_2^+ = D' \leq D$. The meromorphic extension of f to $\mathbb{P}^2$ is given by $[T^D f_1(Z/T, W/T) : T^D f_2(Z/T, W/T) : T^D]$ and the restriction of f to $L_\infty = (T = 0)$ is $f_\infty[Z : W] = [f_1^+(Z,W) : 0^{D-D'} f_2^+(Z,W)]$. Thus, in order to have f_∞ not constant, we need $D = D'$ and f_1^+ not proportional to f_2^+ (otherwise, f sends L_∞ to $[1:0:0]$ or $[1:\lambda:0]$).

The indeterminacy set $I(f)$ of f is the common zeros of f_1^+ and f_2^+: if the line $\mathcal{D}$ of equation $a_j z - b_j w = 0$ satisfies $f_1^+(\mathcal{D}) = \{0\}$ and $f_2^+(\mathcal{D}) = \{0\}$ then $[b_j : a_j : 0]$ is in $I(f)$.

One deduces from above that all the mappings of $\mathcal{G}$ can be written as:

$$\begin{aligned} f(z,w) \quad = \quad & \Big(\prod_{j \leq m} (a_j z - b_j w)^{\alpha_j} P_1(z,w) + Q_1(z,w), \\ & \prod_{j \leq m} (a_j z - b_j w)^{\alpha_j} P_2(z,w) + Q_2(z,w) \Big), \end{aligned} \tag{1}$$

where the a_j and b_j are complex numbers satisfying $(a_j, b_j) \neq (0,0)$, m and the α_j are positive integers, P_1 and P_2 are homogeneous polynomials of degree $d' \geq 1$ with no common factor and the Q_j are polynomials of degree strictly smaller than the degree of f. We denote by d the sum $\sum_{j \leq m} \alpha_j$, so that f is of degree $d + d' = D$.

We have the following formula for the extension of f to $\mathbb{P}^2$:

$$\begin{aligned} f([Z:W:T]) \quad = \quad & \Big[\prod_{j \leq m} (a_j Z - b_j W)^{\alpha_j} P_1(Z,W) + T^D Q_1(\frac{Z}{T}, \frac{W}{T}) : \\ & \prod_{j \leq m} (a_j Z - b_j W)^{\alpha_j} P_2(Z,W) + T^D Q_2(\frac{Z}{T}, \frac{W}{T}) : T^D \ \Big] \end{aligned}$$

thus $f_\infty([Z:W]) = [P_1(Z,W) : P_2(Z,W)]$. Recall that the multiplicity of an indeterminacy point I is the intersection multiplicity at I of L_∞ and $f^{-1}(L)$ where

L is a generic line. The indeterminacy points of f are the $I_j = [b_j : a_j : 0]$ with multiplicity α_j. We assume of course that the (a_j, b_j) are not proportional.

The following proposition shows that we can find f with any given set of indeterminacy points with multiplicity and any given restriction at infinity. Furthermore, it shows that for $D \geq 3$, $\mathcal{G}$ correponds to a Zariski open set of the space of parameters of (1). If $d = d' = 1$, we may have to multiply f satisfying the criterion below by a large enough constant in order to have that the infinity is attracting.

Proposition 2.4 *Let $f = (f_1, f_2)$ be as in (1). Assume that the polynomial $\Phi = f_1 P_2 - f_2 P_1$ is of degree $\geq 2 + d'$. If for all j, $a_j z - b_j w$ does not divide the homogeneous part of maximal degree of Φ, then L_∞ is f-attracting.*

Proof. Let f be as above and N be a small neighborhood of infinity. Observe that for any neighboorhood V of $I(f)$, there exists a constant C such that if $p = (z, w) \in N \backslash V$ we have $C\|p\|^D \leq \|f(p)\|$. So we just have to prove the estimate on V. Since P_1 and P_2 have no common factor, there is $\lambda > 0$ such that $\max(|P_1(z,w)|, |P_2(z,w)|) \leq \lambda \|(z,w)\|^{d'}$ on N. The hypothesis implies that $|\Phi(z,w)| \gtrsim \|(z,w)\|^{\deg \Phi}$ near $I(f)$, hence:

$$2\|f(z,w)\| \geq \frac{|\Phi(z,w)|}{\max(|P_1(z,w)|, |P_2(z,w)|)} \gtrsim \|(z,w)\|^2.$$

The proposition follows. $\square$

Observe that for a generic map $g \in \mathcal{G}$, we have $\deg \Phi = 2d' + d - 1$. The criterion is not optimal, but it is generic for $D \geq 3$ and easy to check. If $D = 2$, we obtain in the same way that $\|f(z,w)\| \gtrsim \|(z,w)\|$.

We use the notation of (1) in the following proposition.

Proposition 2.5 *Let f and g be in $\mathcal{G}$ then $f \circ g \in \mathcal{G}$. More precisely, if $f = (PQ_1 + R_1, PQ_2 + R_2)$ is of degree D and $g = (P'Q'_1 + R'_1, P'Q'_2 + R'_2)$ is of degree D' then $f \circ g = (P''Q''_1 + R''_1, P''Q''_2 + R''_2)$ where:*

$$P'' = (P')^D P(Q'_1, Q'_2), \quad Q''_1 = Q_1(Q'_1, Q'_2), \quad \textit{and} \quad Q''_2 = Q_2(Q'_1, Q'_2).$$

In particular, $f \circ g$ is of degree $D + D'$.

Proof. With the above notations, the homogeneous part of maximal degree of the components of $f \circ g$ are equal to:

$$P(P'Q'_1, P'Q'_2)Q_1(P'Q'_1, P'Q'_2) = (P')^D P(Q'_1, Q'_2)Q_1(Q'_1, Q'_2)$$

and

$$P(P'Q'_1, P'Q'_2)Q_2(P'Q'_1, P'Q'_2) = (P')^D P(Q'_1, Q'_2)Q_2(Q'_1, Q'_2).$$

We only have to check that $Q_1(Q'_1, Q'_2)$ and $Q_2(Q'_1, Q'_2)$ have no common factor: else, since two homogeneous polynomials have no common factor if and only if they

have no common non trivial zero and since Q_1 and Q_2 have no common factor, we would have that Q_1' and Q_2' have a non trivial common zero. □

Recall that a meromorphic mapping $f : \mathbb{P}^2 \to \mathbb{P}^2$ is said to be *algebraically stable* if no algebraic curve is sent to an indeterminacy point after some iterations, equivalently, if f is of algebraic degree D then f^n is of degree D^n for all $n \geq 1$. For such maps, the *Green function* and the *Green current* are defined by $G(p) = \lim_{n\to\infty}(\frac{1}{D^n}\log^+\|f^n(p)\|)$ and $T = dd^cG$ (see [Sib99]). Moreover, if S is a smooth positive closed (1,1)-current of mass 1 on $\mathbb{P}^2$ then $\frac{1}{D^n}(f^n)^*(S) \to T$ in the sense of currents. We deduce the important following corollary from the previous proposition:

Corollary 2.6 *Let $f \in \mathcal{G}$ and $n \in \mathbb{N}$, then $\deg f^n = D^n$ so f is algebraically stable. Furthermore, $(f_\infty)^n = (f^n)_\infty$.*

2.3 Multiplicity of the indeterminacy points, trace of the Green current at infinity

Let E denote the set $\bigcup_{n\geq 0} f^{-n}(I(f)) = \bigcup_{n\geq 0} I(f^n)$. For $p \in E$, we denote by $\lambda_{p,n}$ the real number equal to the multiplicity at p of f^n as an indeterminacy point divided by D^n, that is: $\lambda_{p,n} = \frac{\text{mult}_p(f^n)}{D^n}$ (these numbers will appear in the symbolic dynamics of f). We have the following lemma:

Lemma 2.7 *For all $p \in E$, $(\lambda_{p,n})$ is an increasing sequence bounded by 1. Let λ_p be its limit. Then:*

$$\sum_{p\in E} \lambda_p = 1.$$

Proof. Write $f^n = (P_nQ_{1,n} + R_{1,n}, P_nQ_{2,n} + R_{2,n})$. Recall that $I(f^n)$ is the intersection of L_∞ with the zero set of P_n. By Proposition 2.5:

$$P_{n+1} = (P_n)^D P(Q_{1,n}, Q_{2,n}).$$

Hence, $(\lambda_{p,n})$ is increasing since $(P_n)^D$ is a factor of P_{n+1}.

Set $d_n = \deg(P_n)$ and $d_n' = \deg(Q_{i,n})$. We deduce from Proposition 2.5:

$$d_n' = (d')^n \quad \text{and} \quad d_n = D^n - (d')^n.$$

So, $\sum_{p\in E} \lambda_{p,n} = \frac{d_n}{D^n} \to 1$. This completes the proof. □

Remarks

1. In a way, the indeterminacy points of f^n take asymptotically all the available degree, so they carry the main part of the dynamics near L_∞ (cf. Proposition 2.8).

2. The sequence $(\lambda_{p,n})_n$ can be strictly increasing as we will see in the last two examples of Section 3.6. One can check that $(\lambda_{p,n})_n$ is strictly increasing after some rank if and only if p is preperiodic.

Recall that, on $\mathbb{C}^2$, for f in $\mathcal{G}$ of degree D, the sequence of positive functions $(u_n = \frac{1}{D^n}\log^+\|f^n(z,w)\|)$ almost decreases (i.e. $(u_n + c_n)_n$ is decreasing for some sequence of constant $(c_n)_n$ decreasing to zero) to the *Green function* u of f which is a potential of the Green current T. Furthermore, the function $\tilde{u}(z,w) = u(z,w) - \frac{1}{2}\log(|z|^2+|w|^2+1)$ is a bounded quasi-plurisubharmonic function on $\mathbb{C}^2$, thus it extends to $\mathbb{P}^2$, and this extension satisfies $dd^c\tilde{u} = T - \omega_{FS}$ where ω_{FS} is the Fubini-Study form on $\mathbb{P}^2$.

We will see in Proposition 2.8 that $\tilde{u}_{|L_\infty}$ is not identically equal to $-\infty$ so we can define the measure $m_\infty = T \wedge [L_\infty]$ which is the trace of the Green current at infinity. Since the sequence of functions $\tilde{u}_n(z,w) := u_n(z,w) - \frac{1}{2}\log(|z|^2+|w|^2+1)$ is almost decreasing, m_∞ is the limit in the sense of current of the sequence $((dd^c\tilde{u}_n(z,w)+\omega_{FS})\wedge[L_\infty])$. In particular, we have $m_\infty = dd^c(\tilde{u}_{|L_\infty}) + (\omega_{FS})_{|L_\infty}$. The next proposition shows that m_∞ is a combination of Dirac masses at the points of E, with computable coefficients.

Proposition 2.8 *Let f be in $\mathcal{G}$ and $\tilde{u}$ be as above. For $p \in E$, we denote by $[a_p : b_p : 0]$ its homogeneous coordinates. Then:*

$$\tilde{u}([z:w:0]) = \log^+\Big(\prod_{p\in E}|a_p w - b_p z|^{\lambda_p}\Big) - \frac{1}{2}\log(|z|^2+|w|^2).$$

In particular, we have the formula:

$$m_\infty = \sum_{p\in E}\lambda_p\delta_p$$

where δ_p is the Dirac mass at p.

Proof. With the above notations, we have that in $\mathbb{C}^2$:

$$\begin{aligned}\tilde{u}_n(z,w) &= \frac{1}{D^n}\log^+\|(P_nQ_{1,n}+R_{1,n})(z,w),(P_nQ_{2,n}+R_{2,n})(z,w)\| \\ &\quad -\frac{1}{2}\log(|z|^2+|w|^2+1).\end{aligned}$$

So, first apart from the point of E, and hence everywhere on L_∞ by semi-continuity, the extension is given by:

$$\tilde{u}_n([z:w:0]) = \frac{1}{D^n}\log\|(P_nQ_{1,n})(z,w),(P_nQ_{2,n})(z,w)\| - \frac{1}{2}\log(|z|^2+|w|^2)$$

By definition of the $\lambda_{p,n}$ and Corollary 2.6, there is a constant C_n depending on the choice of the coordinates of the elements of E such that:

$$\tilde{u}_n([z:w:0]) = \sum_{p\in E}\lambda_{p,n}\log|a_pw-b_pz| + \frac{1}{D^n}\log\|f_\infty^n[z:w]\| + C_n - \frac{1}{2}\log(|z|^2+|w|^2).$$

From one-dimensional theory, we know that $\frac{1}{d'^n}\log|(f_\infty)^n[z:w]-\frac{1}{2}\log(|z|^2+|w|^2)$ converges to a continuous function on L_∞ and $\sum_{p\in E}\lambda_{p,n}\log(|a_p w - b_p z|)$ converges thanks to the previous lemma. The last identity and the fact that $d' < D$ imply the first formula in the proposition. The formula giving m_∞ is then clear by the Poincaré formula. $\square$

Remark. The previous proof can be applied to all the algebraically stable polynomial maps of $\mathbb{C}^2$ with indeterminacy points on L_∞.

2.4 Topological degree

Let N be a small enough neighborhood of L_∞ and V be a neighboorhood of $I(f)$, then there are constants C and C' such that for p in $N\backslash V$, we have:

$$C\|p\|^D \leq \|f(p)\| \leq C'\|p\|^D.$$

Let us assume here that the considered mapping satisfies in addition: for all $I \in I(f)$, there exist a number l_I, a neighborhood $V(I)$ of I, a neighborhood $V(f_\infty(I))$ of $f_\infty(I)$ and constants C_1 and C_2 such that for all $p \in V(I)$ with $f(p) \notin V(f_\infty(I))$, we have:

$$C_1\|p\|^{l_I} \leq \|f(p)\| \leq C_2\|p\|^{l_I} \qquad (2)$$

This condition is easy to check in practice. Under these assumptions, we can compute the topological degree of f which is the mass of the pull-back of any probability measure by f. The difference with the case with no dynamics on L_∞ is that we have to count the number of preimages of a generic line by f_∞. We have the following proposition:

Proposition 2.9 *Let $f \in \mathcal{G}$ satisfying (2). Then the topological degree of f is given by:*

$$d_t = \sum_{I\in I(f)} l_I\alpha_I + d'D.$$

In particular, we have $d_t > D$.

Proof. Let L be a generic line, we consider the probability measure $[L_\infty]\wedge[L]$ (which is the Dirac mass at the intersection of L and L_∞). By definition, its pull back by f is of mass d_t. After some change of coordinates, we can assume that the point $[1:0:0]$ is not on L and $f^{-1}(L)$. So we work in the coordinates $(u,v) = (Z/W, T/W)$ where a potential of $L_\infty = (v=0)$ is $\varphi(u,v) = \log|v|$. We must compute:

$$\int_{\mathbb{P}^2} f^*([L_\infty]\wedge[L]) = \int_{f^{-1}(L)} dd^c(\varphi\circ f).$$

For each I in $I(f)$, let $\mathbb{B}_I$ be a bidisk in $V(I)$ for the (u,v) coordinates, and for each p in $f_\infty^{-1}(L_\infty\cap L)$ let $\mathbb{B}_p$ be a bidisk around p. Since L is a generic line, we

can assume that $f_\infty^{-1}(L_\infty \cap L) \cap I(f) = \varnothing$ and that all those bidisks are disjoint. The previous integral become:

$$d_t = \sum_{I \in I(f)} \int_{f^{-1}(L) \cap \mathbb{B}_I} dd^c(\varphi \circ f) + \sum_{p \in f_\infty^{-1}(L_\infty \cap L)} \int_{f^{-1}(L) \cap \mathbb{B}_p} dd^c(\varphi \circ f).$$

Observe that $\varphi \circ f - l_I \log|v|$ is a bounded pluriharmonic function on $\mathbb{B}_I \backslash L_\infty$ thanks to (2), so it defines in fact a pluriharmonic function on $\mathbb{B}_I$. Hence, on these bidisks, $dd^c(\varphi \circ f)$ is equal to l_I times the current of integration on L_∞. Using the same argument for $\mathbb{B}_p$, we deduce:

$$d_t = \sum_{I \in I(f)} \int_{f^{-1}(L) \cap \mathbb{B}_I} l_I dd^c(\log|v|) + \sum_{p \in f_\infty^{-1}(L_\infty \cap L)} \int_{f^{-1}(L) \cap \mathbb{B}_p} D dd^c(\log|v|)$$

which is what we wanted since $\int_{f^{-1}(L) \cap \mathbb{B}_I} dd^c(\log|v|) = \alpha_I$ is the intersection multiplicity at I of $L\infty$ and $f^{-1}(L)$ and since there are d' preimages of $L \cap L_\infty$ by f_∞. $\square$

3 Structure of the Julia set and of the Green current near infinity

We assume in this section that f_∞ is uniformly hyperbolic (i.e. the forward orbit of each critical point converges towards some attracting periodic orbit), and that the indeterminacy points are not in the Julia set J_∞ of f_∞. After a unitary change of coordinates, we can assume that $[1:0:0]$ is not in $J_\infty \cup E$. Hence $(u, v) = (Z/W, T/W)$ is a coordinate system of a neighborhood of $L_\infty \backslash [1:0:0]$ where $L_\infty = (v = 0)$. We also need the hypothesis that the indeterminacy points are not periodic.

We construct suitable boxes (polydisks) around the elements of E such that f defines horizontal-like maps between these boxes.

After decomposing the Julia set into pieces near infinity, we construct a subshift on $E^{\mathbb{N}}$. We then decompose the Green current along these pieces by pulling-back a smooth vertical positive closed $(1,1)$-form in the boxes which gives the Green current in a neighborhood of infinity. Observe that the set of maps we consider contains an open set in the space of parameters.

Next, we give an application for the escape rate of f and we explain how to obtain a weaker decomposition in the more general case where some indeterminacy points are in J_∞. Finally we explain our results through examples.

3.1 Construction of the boxes

The purpose of this section is to prove the following proposition:

Proposition 3.1 *For all p in E, there is a bidisk Δ_p centered at p such that f induces by restriction a horizontal-like map from Δ_p to Δ_q for all $q \in E$ if $p \in I(f)$ and for $q = f_\infty(p)$ if p is not an indeterminacy point. We denote by $f_{p,q}$ this restriction.*

The bidisks can be taken arbitrarily small. We can choose them so that for all $I \in I(f)$ and all $q \in E - \{I\}$ then $\Delta_I \cap \Delta_q = \varnothing$, and for all $p \in E$ and all $q, q' \in f_\infty^{-1}(p)$ then $\Delta_q \cap \Delta_{q'} = \varnothing$.

Since f_∞ is uniformly hyperbolic, we can put a smooth conformal metric g on L_∞ such that $\|Df_\infty(z)\|_g \geq \lambda > 1$ on J_∞. Let us remark that E is discrete in the Fatou set $F_\infty := L_\infty \backslash J_\infty$ since the only components of F_∞ are basins of attraction of periodic cycles and $\overline{E} = E \cup J_\infty$ (see [Mil99]). The idea is first to construct disks on L_∞ which will be thickened to get bidisks. So, we use the following lemma:

Lemma 3.2 *There is a constant $c > 0$ such that for all p in E, there exists a disk $\mathbb{D}_p$ for the metric g such that if $p \in I(f)$ and $q \in E$ then $dist_g(f_\infty(\partial \mathbb{D}_p), \mathbb{D}_q) \geq c$ and if $p \in E \backslash I(f)$ then $dist_g(f_\infty(\partial \mathbb{D}_p), \mathbb{D}_{f_\infty(p)}) \geq c$. Furthermore, we can choose the radii of those disks to be bounded and arbitrarily small.*

Proof. Let U be an open neighborhood of J_∞ in L_∞ with smooth boundary such that $\|Df_\infty(z)\|_g \geq \rho > 1$ on U and $f_\infty^{-1}U \subset U$. There is only a finite number of elements of E in $L_\infty \backslash U$. Modifying U if necessary, we can assume that $I(f) \cap U = \varnothing$ and $\partial U \cap E = \varnothing$.

For $I \in I(f)$ such that $f_\infty(I) \notin E$, we consider $\mathbb{D}_I$ a disk centered at I on L_∞ for the metric g with $f_\infty(\mathbb{D}_I)$ far from the other points of E. Restricting $\mathbb{D}_I$ if necessary, we can assume that for all p in $f_\infty^{-1}\{I\}$ there is a disk $\mathbb{D}_p$ centered in p on L_∞ such that $f_\infty(\partial \mathbb{D}_p) \cap \mathbb{D}_I = \varnothing$ (we use the fact that f_∞ is open). We iterate this construction with the preimages of all the p till all of them are in U. Of course, we may have to shrink $\mathbb{D}_I$ at each step. We apply this process to all the elements of $I(f)$ such that $f_\infty(I) \notin E$.

Since we assumed the elements of $I(f)$ are not periodic for f_∞, we then have disks $\mathbb{D}_p$ for all the p in $E \backslash U$ such that $f_\infty(\partial \mathbb{D}_p) \cap \mathbb{D}_{f_\infty(p)} = \varnothing$. Let r be the smallest radius of all these disks. It can be chosen arbitrarily small.

Next, by hyperbolicity, there is some $\varepsilon_0 > 0$ such that f_∞ is injective on any disk $\mathbb{D}_g(z, \varepsilon_0)$ for all z in $f_\infty^{-1}U$, and is closed to its differential. Namely, for all $\varepsilon \leq \varepsilon_0$, there is a $\rho' > 1$ such that we have $\mathbb{D}_g(f(z), \rho'\varepsilon) \Subset f_\infty(\mathbb{D}_g(z, \varepsilon))$ for z in $f_\infty^{-1}U$. Then, for r small enough, we have some r' such that for all p in $E \cap f_\infty^{-1}U$, the disk $\mathbb{D}_p = \mathbb{D}_g(p, r')$ satisfies $f_\infty(\partial \mathbb{D}_p) \cap \mathbb{D}_{f_\infty(p)} = \varnothing$. The existence of the constant $c > 0$ is then clear by construction for $p \in L_\infty \backslash f_\infty^{-1}(U)$ and by hyperbolicity for $p \in E \cap f_\infty^{-1}(U)$. □

Proof of Proposition 3.1. Recall that the line at infinity is f-attracting: there is a constant $C > 1$ such that for $M = (z, w)$ in $\mathbb{C}^2$ with $\|M\| \geq A$, we have $\|f(M)\| \geq C\|M\|$ where $\| \star \|$ denotes the euclidean norm. Furthermore:

$$\|M\|^2 = |z|^2 + |w|^2 = |\frac{u}{v}|^2 + |\frac{1}{v}|^2.$$

If $p=(u_p,v_p)$, define $\Delta_p=\mathbb{D}_p\times\{v<\frac{\epsilon}{\sqrt{1+|u_p|^2}}\}$ with ε small. For $M=(u,v)\in\Delta_p$, we have that

$$(1+\nu)^{-1}(|u_p|^2+1)|\frac{1}{v}|^2\leq\|M\|^2\leq(1+\nu)(|u_p|^2+1)|\frac{1}{v}|^2$$

where $\nu>0$ depends only on the radius of $\mathbb{D}_p$ (since u is uniformly bounded) and goes to zero with it. We take the radii of the $\mathbb{D}_p$ small enough so that the square of the norm of an element in Δ_p is close to $(|u_p|^2+1)|\frac{1}{v}|^2$.

We choose ε so that all the Δ_p are in the domain where the infinity is attracting. Restricting r' which is the supremum of the radii of the disks $\mathbb{D}_p$ if necessary, we can assume that $f(\Delta_p)\cap\partial\Delta_q\subset\partial_v\Delta_q$ for all q if p is an indeterminacy point and for $q=f(p)$ otherwise. Now, using the uniform continuity of f in the complement of some small neighborhood of the indeterminacy set and the existence of c in Lemma 3.2, we can choose ε so that $f(\partial_v\Delta_p)\cap\Delta_q=\varnothing$ for all $q\in E$ if $p\in I(f)$ and for $q=f(p)$ otherwise. Finally, since the image of any small neighborhood of an indeterminacy point by f contains the whole line at infinity, we have $f(\Delta_p)\cap\Delta_q\neq\varnothing$. The last part of the proposition is clear for r' small enough (we use the hyperbolicity of f once again here). $\square$

3.2 Construction of the subshift

Now, we define the symbolic dynamics which will appear in the decomposition of the Green current. First, we will need to know the degree of the horizontal-like maps $(f_{p,q})$. Recall that α_i is the multiplicity of the indeterminacy point $I_i\in I(f)$. We take the notations of (1). The following lemma is clear:

Lemma 3.3

1. *If p is in $E\backslash I(f)$, then the degree of $f_{p,q}$ is the local degree of f_∞ at p,*
2. *if $p=I_j$ is in $I(f)$ and $q\neq f_\infty(I_j)$, then the degree of $f_{p,q}$ is α_j,*
3. *if $p=I_j$ is in $I(f)$ and $q=f_\infty(I_j)$, then the degree of $f_{I_j,f_\infty(I_j)}$ is the sum of α_j and the local degree of f_∞ at I_j.*

Define $\Sigma'=E^{\mathbb{N}}$ and $\Sigma=\{(\alpha_n)\in\Sigma',f_{u_n,u_{n+1}}\text{ exists}\}$, the space of itineraries between indeterminacy points and theirs preimages. We consider the left shift σ on Σ and Σ'. Define $N=\bigcup_{p\in E}\Delta_p$. For $\alpha\in\Sigma$, let us consider:

$$\mathcal{K}_\alpha=\{p\in N,\ f^j(p)\in\Delta_{\alpha(j)}\}.$$

Then, for all $\alpha\in\Sigma$, $\overline{\mathcal{K}_\alpha}$ is not empty as a decreasing intersection of vertical closed sets in $\Delta_{\alpha(0)}$. Let $\mathcal{K}$ be the union of all the $\mathcal{K}_\alpha$ so that $\mathcal{K}\subset N$.

Observe that $T \wedge [L_\infty]$ is the slice of T by $(v = 0)$. Using the formula giving the trace of T on L_∞ and the invariance of T $(f^*T = DT)$, we have that:

$$\forall p \in E, \lambda_p = \frac{1}{D} \sum_{q \in E} d_{p,q} \lambda_q$$

with the convention that $d_{p,q} = 0$ if $f_{p,q}$ is not defined. For all $p \in E$, we deduce:

$$1 = \sum_{q \in E} \frac{d_{p,q} \lambda_q}{D \lambda_p}. \tag{3}$$

For example, if p is not an indeterminacy point, we have that all the $d_{p,q}$ are zero except for $q = f(p)$ and the formula becomes:

$$1 = \frac{d_{p,f(p)} \lambda_{f(p)}}{D \lambda_p}.$$

And if $p = I$ is in the indeterminacy set with $d_{I,q}$ constant (i.e. $f_\infty(I) \notin E$), then:

$$1 = \sum_{q \in E} \lambda_q.$$

Let $A := (a_p^q)_{p,q \in E}$ be the infinite matrix defined by $a_p^q = \frac{d_{p,q} \lambda_q}{D \lambda_p}$. The entry a_q^p can be seen as the *probability to go from* Δ_p *to* Δ_q *by* f in term of slice mass (see the proof of Theorem 3.5). Of course, if p is not an indeterminacy point, one always goes to $\Delta_{f(p)}$ (the probability is 1). We put on Σ the Borel measure ν defined by:

$$\nu(\{\alpha \in \Sigma,\ \alpha(0) = \alpha_0, \ldots, \alpha(n) = \alpha_n\}) = \lambda_{\alpha_0} \times \prod_{i=0}^{n-1} a_{\alpha_i}^{\alpha_i + 1} = \lambda_{\alpha_n} \times \prod_{i=0}^{n-1} \frac{d_{\alpha_i, \alpha_{i+1}}}{D}.$$

Proposition 3.4 *The left shift σ on Σ defines a subshift for which the measure ν is invariant and mixing.*

Proof. Definitions and facts on symbolic dynamics and especially subshift can be found in [KH95], pp. 156-158. By (3), we already have for all $p \in E$ that

$$\sum_q a_p^q = 1 \tag{4}$$

What remains to be proved is that the vector (λ_p) is an eigenvector for the matrix ${}^tA$ associated with the eigenvalue 1 (that gives the invariance of ν). That is:

$$\sum_p a_p^q \lambda_p = \lambda_q \tag{5}$$

which is clear again by (3).

Furthermore, the matrix A is transitive in the sense that for each (p, q) the entry of index (p, q) in A^n is stricly positive for some n (it is clear if p is in the indeterminacy set for $n = 1$ and if $p \in f^{-j}(I(f))$, then it is true for $n = j + 1$). We deduce that ν is mixing. □

Remark. We can consider only a finite part of E containing the indeterminacy points and their preimages up to some order and regroup the rest of the elements of E in a same box. Then we can obtain a finite Markov chain, but we lose some part of the information.

3.3 Decomposition of the Green current

Let us denote by $\mathcal{L}_{p,q}$ the operator $\frac{1}{d_{p,q}} f^*_{p,q}$ acting on vertical currents. Recall that f is a polynomial map of $\mathbb{C}^2$ having indeterminacy points on L_∞ which is f-attracting. The map f_∞ is hyperbolic and the indeterminacy points of f are on the Fatou set of f_∞ and not periodic. We can now prove our main theorem:

Theorem 3.5 *1. There exists an at most countable set $\Theta \subset \Sigma$ such that for all $\alpha \in \Sigma \backslash \Theta$, there is a unique current T_α satisfying the following property: for all sequence of currents (S_{k+1}) of bidegree (1,1), positive, closed, vertical in $\Delta_{\alpha(k+1)}$ of slice mass 1, we have:*

$$\mathcal{L}_{\alpha(0),\alpha(1)} \dots \mathcal{L}_{\alpha(k),\alpha(k+1)} S_{k+1} \to T_\alpha.$$

2. The Green current T admits the following decomposition in N:

$$T = \int_\Sigma T_\alpha d\nu(\alpha).$$

Proof. Since $\mathcal{K} = \bigcup \mathcal{K}_\alpha$, only a countable number of $\mathcal{K}_\alpha$ have positive Lebesgues measure. Then Theorem 2.3 implies the first part.

For the second part, let S'_p be a smooth positive closed (1,1)-form in $\mathbb{P}^2$ such that near L_∞, S'_p has its support in $\Delta'_p = \mathbb{D}'_p \times \{v < \frac{\epsilon}{\sqrt{1+|u_p|^2}}\}$, with $\mathbb{D}'_p \Subset \mathbb{D}_p$. Let S_p be the restriction of S'_p to Δ_p, it is a vertical positive closed current. Normalize S'_p so that S_p is of slice mass λ_p. Observe that if $S' = \sum S'_p$ then $\lim \frac{1}{D^n}(f^n)^*(S') = \sum \lim \frac{1}{D^n}(f^n)^*(S'_p) = T$ since $\sum \lambda_p = 1$. Define $S = \sum S_p$. Finally, write:

$$\Sigma_n = \{(a_0, a_1, \dots, a_{n-1}) \in E^n | \exists \alpha \in \Sigma, \forall i \leq n-1,\ a_i = \alpha(i)\}$$

and for $a \in \Sigma_n$, write C_a for the cylinder:

$$\{\alpha \in \Sigma |\ \alpha(0) = a_0, \dots, \alpha(n-1) = a_{n-1}\}.$$

Pulling back S by f gives:

$$\begin{aligned} \frac{1}{D}f^*S &= \frac{1}{D}\sum_{p,q\in E} f^*_{p,q}S_q \\ &= \sum_{\alpha\in\Sigma_1}\frac{d_{\alpha(0),\alpha(1)}}{D}\mathcal{L}_{\alpha(0),\alpha(1)}S_{\alpha(1)}. \end{aligned}$$

We iterate:

$$\begin{aligned} \frac{1}{D^k}(f^k)^*S &= \sum_{\alpha\in\Sigma_k}\frac{\prod_{i=0}^{k-1} d_{\alpha(i),\alpha(i+1)}}{D^k}\mathcal{L}_{\alpha(0),\alpha(1)}\dots\mathcal{L}_{\alpha(k-1),\alpha(k)}S_{\alpha(k)} \\ &= \sum_{\alpha\in\Sigma_k}\nu(C_\alpha)\mathcal{L}_{\alpha(0),\alpha(1)}\dots\mathcal{L}_{\alpha(k-1),\alpha(k)}\frac{S_{\alpha(k)}}{\lambda_k}. \end{aligned}$$

The left hand side goes to the Green current T. By the first part of the theorem, the general term of the right hand side tends to T_α for α generic so we get the result by dominated convergence. $\square$

Remark. The dynamics of f near infinity is semi-conjugated to the subshift σ in the sense that $f(\mathcal{K}_\alpha)\subset\mathcal{K}_{\sigma(\alpha)}$.
We see that the current is carried by $\mathcal{K}$ which does not meet J_∞. So, as announced in the introduction, the local stable manifolds to the Julia set of f_∞ do not carry any part of the Green current, but they are contained in its support.

3.4 Escape rate

We take $f\in\mathcal{G}$ satisfying the condition (2), we also suppose that $f_\infty(I(f))\cap E=\varnothing$ (else, we would have p in E such that $f(p)\in f(V(I(f)))$). We want to compute the possible values of the *upper escape rate* $\bar{l}$ where $\log(\bar{l}) = \limsup\frac{1}{n}\log^+\log^+\|f^n\|$ which becomes $\limsup\frac{1}{n}\log\log(\|f^n\|)$ in N. In the same way, we define the *lower escape rate* $\underline{l}$ and we are interested in knowing where these two functions match up, in which case we note l their common value which we simply call the *escape rate*.

For $p\in E\backslash I(f)$, we set $l_p = D$. We have the following lemma:

Lemma 3.6 *Let $\alpha\in\Sigma$ and $q\in\mathcal{K}_\alpha$. We have:*

$$\frac{1}{n}\log\log\|f^n(q)\| = \frac{1}{n}\log(l_{\alpha(0)}l_{\alpha(1)}\dots l_{\alpha(n-1)}) + O(\frac{\log n}{n}).$$

Proof We have constants c_1 and c_2 such that:

$$c_1\leq\log\|f^{j+1}(q)\| - l_{\alpha(j)}\log\|f^j(q)\|\leq c_2.$$

Taking a combination of these inequalities for $j\leq n-1$ gives:

$$c_1\Big(\sum_{j=0}^{n-1} l_{\alpha(j+1)}\dots l_{\alpha(n-1)}\Big)+l_{\alpha(0)}\dots l_{\alpha(n-1)}\log\|q\|\le\log\|f^n(q)\|,$$

with a similar inequality for the right hand side. Taking the logarithm and dividing by n give:

$$\Big|\frac{1}{n}\log\log\|f^n(q)\|-\frac{1}{n}\log(l_{\alpha(0)}\dots l_{\alpha(n-1)})\Big|\le\frac{1}{n}\log\Big(\log\|q\|+C\sum_{j=0}^{n-1}\frac{1}{l_{\alpha(0)}\dots l_{\alpha(j-1)}}\Big)$$

The sum in the right hand side is a $O(n)$ which concludes the proof. □

Choosing a suitable α, we deduce from the lemma that the range of the escape rate in N is $[\min l_I, D]$ (the details are left to the reader). In this case, it is interesting to observe that the set of possible escape rates is an interval which is a new property for polynomial mappings. Let λ denote the slice mass $1-\sum_{I\in I(f)}\lambda_I$ of T outside a neighborhood of $I(f)$. We have the following theorem:

Theorem 3.7 *For $\|T\|$-almost every point q in N, the escape rate $l(q)$ exists and is equal to $D^\lambda\prod_{I\in I(f)} l_I^{\lambda_I}$.*

Proof. Since the left shift σ is ergodic for ν, the Birkhoff's ergodic theorem yields that for ν-almost every α:

$$\exp\Big(\frac{1}{n}\sum_{i=0}^{n-1}\log l_{\sigma^i(\alpha)(0)}\Big)\to\exp\Big(\int_\Sigma\log l_{\alpha(0)}d\nu\Big)=D^\lambda\prod_{I\in I(f)} l_I^{\lambda_I}.$$

And the theorem follows from the previous lemma and Theorem 3.5. □

3.5 Generalization

In the case where some indeterminacy points are on J_∞ (possibly periodic), we can obtain a decomposition of the Green current by building a cover of J_∞ by disks such that for all $\mathbb{D}$ in this cover, there exist disjoint disks $\mathbb{D}_1,\mathbb{D}_2,\dots,\mathbb{D}_{d'}$ in the cover with $f_\infty^{-1}(\mathbb{D})\subset\mathbb{D}_1\cup\mathbb{D}_2\cup\dots\cup\mathbb{D}_{d'}$ and $\mathbb{D}\Subset f_\infty(\mathbb{D}_i)$ for all $i\le d'$. The trick is to have two disks $\mathbb{D}_I$ and $\mathbb{D}'_I$ around each indeterminacy point $I\in I(f)$ so that $\partial f_\infty(\mathbb{D}_I)\cap\overline{\mathbb{D}}=\varnothing$ or $\partial f_\infty(\mathbb{D}'_I)\cap\overline{\mathbb{D}}=\varnothing$. Finally, we follow the construction of Section 3.1 with U being replaced by the union of all those disks.

This time we only have a finite number of bidisks and when we pull back the Green current near some point of E to an indeterminacy point I in J_∞, we may have to choose between the two bidisks centered at I in order to have a horizontal-like map. We only get a finite subshift, but taking a finer cover, we get more precision on the decomposition (only on a smaller neighborhood of L_∞). Somehow the decomposition is not intrinsic because we do not pull back according to the itinerary but it assures that the Green current is not extremal in a neighborhood of L_∞.

3.6 Examples

First let us explain our results in two examples where the dynamics at infinity is linear.

Example 1. Consider the case where f_∞ is given by $u \mapsto 2u$ and where the indeterminacy set is reduced to $(1,0)$ with multiplicity 1 in the (u,v) coordinates (thanks to Proposition 2.4, we know this case exists, take for example $f(z,w) = C(2z(z-w)+z, w(z-w))$ for C large enough). Then, using Proposition 2.8, we find that:

- $E = \{p_n = (\frac{1}{2^n}, 0)\ n \geq 0\}$.
- $\lambda_n = \frac{1}{2^n}$
- the matrix of the subshift is:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

An element $\alpha \in \Sigma$ can be written $(p_{n_1}, p_{n_1-1}, \dots, p_0, p_{n_2}, \dots, p_0, \dots)$ for some sequence (n_i) in $\mathbb{N}$. The dynamics in the space of itineraries is simple: a point in $\mathcal{K}_\alpha$ where $\alpha_0 = p_{n_1}$ is sent near p_{n_1-1} then near p_{n_1-2} and so on untill it arrives near p_0, in which case it can be sent near any element of E since p_0 is an indeterminacy point.

Example 2. This time, we still take f_∞ given by $u \mapsto 2u$ and we suppose that the indeterminacy points are $I_0 = (2,0)$ and $I_1 = (1,0)$ with multiplicity 1 in the (u,v) coordinates, so $D = 3$ (for example: $f(z,w) = (2z(z-w)(z-2w)+z^2, w(z-w)(z-2w)))$. In this case, we have that $f_\infty^{-1} I_0 = I_1$. Again, using Proposition 2.8, we find that:

- $E = \{p_n = (\frac{1}{2^{n-1}}, 0)\ n \geq 0\}$.
- We have $\lambda_0 = \lambda_{I_0} = \frac{1}{3}$, $\lambda_1 = \lambda_{I_1} = \frac{4}{9}$, $\lambda_{p_n} = \frac{4}{2^{n+1}}$.
- The matrix of the subshift is:

$$\begin{pmatrix} \frac{1}{3} & \frac{4}{9} & \frac{4}{27} & \frac{4}{3^4} & \dots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{9} & \frac{1}{3^3} & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The interesting fact here is that the entries of the second row are not proportionnal to the the slice mass, indeed a point near I_1 will have "more chances" to be sent on Δ_{I_0} by f since $f_\infty(I_1) = I_0$.

Now, we consider the case were the indeterminacy points are in the exceptionnal set of f_∞ (namely $f^{-1}(I) = I$). Observe that this case does not satysfy the hypothesis of Theorem 3.5 since the indeterminacy points are periodic.

Example 3. The method used to treat the example easily extends to the case where f_∞ admits a totally invariant point (i.e. $f_\infty^{-1}(e) = e$) which is equal to the indeterminacy set but for the sake of simplicity,we only consider:

$$f : (z, w) \mapsto (z^3 + w^2, zw^2).$$

By Proposition 2.4, L_∞ is f-attracting. We even have $\|f(z, w)\| \geq \|(z, w)\|^2$ for $\|(z, w)\|$ large enough. The meromorphic extension of f to $\mathbb{P}^2$ is given by: $f([Z : W : T]) = [Z^3+TW^2 : ZW^2 : T^3]$. The indeterminacy set of f is reduced to $I_0 = [0 : 1 : 0]$ and the dynamics at infinity is given by $f_\infty : [z : w : 0] \mapsto [z^2 : w^2 : 0]$ (so $f_\infty^{-1}(I_0) = I_0$). Thus f is in $\mathcal{G}$ and is algebraically stable.

The topological degree d_t of f, which is by definition the number of preimages of a generic point, is equal to 8 (solve $f(z, w) = (0, 1)$). It is greater than the algebraic degree.

We use the coordinates $(u, v) = (\frac{Z}{W}, \frac{T}{W})$ in which L_∞ is given by $(v = 0)$. The map f becomes:

$$f : (u, v) \mapsto (\frac{u^3 + v}{u}, \frac{v^3}{u}).$$

In these coordinates, the point I_0 becomes $(0, 0)$. The map f_∞ is given by $u \mapsto u^2$ for which the Julia set J_∞ is the unit circle $(|u| = 1)$. We have the following lemma:

Lemma 3.8 *Let $V = \{(u, v),\ |u| < \frac{1}{2}$ and $|v| < \frac{1}{4}|u|^3\}$, then $f(V) \subset V$.*

Proof. Let (u, v) be in V. We check:

$$\frac{|u^3 + v|}{|u|} \leq |u|^2 + \frac{|v|}{|u|} < \frac{1}{4} + \frac{|u|^2}{4} < \frac{1}{2}$$

We also have the inequalities:

$$\begin{aligned} \frac{|u^3 + v|}{|u|} &\geq |u|^2 - \frac{|v|}{|u|} > |u|^2 - \frac{|u|^2}{4} > \frac{1}{2}|u|^2 \\ \frac{|v|^3}{|u|} &< \frac{1}{4^3}|u|^8. \end{aligned}$$

It is then sufficient to check that:

$$\frac{1}{4^3}|u|^8 < \frac{1}{4}(\frac{1}{2}|u|^2)^3$$

which is obvious. $\square$

We deduce from the lemma that V is in the Fatou set since the sequence of iterates is normal there. Let then $\mathbb{D}_0 \subset \mathbb{D}_1$ be disks on L_∞ centered on I_0, small enough to be contained in V, with $f_\infty^{-1}(\mathbb{D}_0) \Subset \mathbb{D}_1$. Let $\mathbb{D}_2$ be a disk centered on $[1:0:0]$ containing the Julia set of f_∞ with $\partial\mathbb{D}_2 \subset V$. We have that $f^{-1}(\mathbb{D}_2) \Subset \mathbb{D}_2$. We can shrink those disks to have $\mathbb{D}_1 \cap \mathbb{D}_2 = \varnothing$.

As in Proposition 3.1, we want to "thicken" those disks in order to have bidisks such that f defines by restriction horizontal-like maps between them. Close to I, the norm of a point (in the (z,w) coordinates) is given by $|v|^{-1}$, but next to $[1:0:0]$, it is controled by $\frac{|u|}{|v|}$ so we use the coordinates $(u',v') = (\frac{T}{Z}, \frac{W}{Z})$ there. Then, we define $\Delta_0 = \mathbb{D}_0 \times (|v| < \varepsilon)$, $\Delta_1 = \mathbb{D}_1 \times (|v| < \varepsilon)$ and $\Delta_2 = \mathbb{D}_2 \times (|u'| < \varepsilon')$. Take ε and ε' small enough so that the vertical boundaries of the bidisks are relatively compact in V. Observe that $\Delta_1 \backslash \Delta_0 \subset V$ is in the Fatou set of f.

Recall that since I_0 is an indeterminacy point, any neighborhood of I_0 is sent on the whole L_∞. Since L_∞ is f-attracting, and by uniform continuity of f away from any neighborhood of I_0, we can chose ε and ε' small enough so that:

- $f : \Delta_1 \to \Delta_0$ defines a horizontal-like map of degree 3 denoted by $f_{1,0}$
- $f : \Delta_1 \to \Delta_2$ defines a horizontal-like map of degree 1 denoted by $f_{1,2}$
- $f : \Delta_2 \to \Delta_2$ defines a horizontal-like map of degree 2 denoted by $f_{2,2}$

Next, we consider the Green current T of f. We know that its support is contained in the Julia set of f (see [Sib99]). So we know that in some neighborhood of infinity, T can be written as $T_1 + T_2$ where T_1 and T_2 are vertical positive closed currents in $\Delta_0 \subset \Delta_1$ and in Δ_2. Pulling-back T_1 and T_2 and using the invariance of T, we see that:

$$\frac{1}{3} f^* T = T = T_1 + T_2$$

So:

$$\begin{aligned} T_1 &= \frac{1}{3} f_{1,0}^* T_1 + \frac{1}{3} f_{1,2}^* T_2 \\ T_2 &= \frac{1}{3} f_{2,2}^* T_2 \end{aligned}$$

Calling m_1 and m_2 the slice masses of T_1 and T_2, we can compute them using the previous equation and the fact that the pull-back of a vertical current of slice mass m by a horizontal-like map of degree d is of slice mass dm. So, we have:

$$\begin{aligned} m_1 &= m_1 + \frac{1}{3} m_2 \\ m_2 &= \frac{2}{3} m_2 \end{aligned}$$

Hence, $m_2 = 0$ and so $T_2 = 0$. In particular, the support of the Green current of f is *strictly contained* in the Julia set J since the stable manifolds associated to the Julia set J_∞ of f_∞ are in J but $\mathrm{supp}(T)$ does not meet J_∞. In [FS95], there is a different example of such phenomenon.
For $\varepsilon > 0$, we consider the small perturbation f_ε defined by:

$$f_\varepsilon : (z, w) \mapsto ((z + \varepsilon w)z^2 + w^2, (z + \varepsilon w)w^2).$$

We check that f_ε gives the same map at infinity than f and that the indeterminacy point is now $I_\varepsilon = [-\varepsilon : 1 : 0]$. We see that the preimages of I_ε accumulate on the Julia set of $f_{\varepsilon\infty}$ and we have seen that they are on the support of the Green current which contrasts with what happens for f.

Example 4. Consider the following families of polynomial maps:

$$\begin{aligned} f(z,w) &= \left(z^{n_1}w^{n_2}z^n + R_1(z,w), z^{n_1}w^{n_2}w^n + R_2(z,w)\right) \\ g(z,w) &= \left(z^{n'_1}w^{n'_2}w^{n'} + R'_1(z,w), z^{n'_1}w^{n'_2}z^{n'} + R'_2(z,w)\right) \end{aligned}$$

where the n_i, n'_i, n and n' are positive integers and the R_i and R'_i are polynomials of degree smaller than $n_1 + n_2 + n$ and $n'_1 + n'_2 + n'$ respectively chosen so that L_∞ is attracting for these polynomial mappings (use Proposition 2.4). The indeterminacy set is then $I = I(f) = I(g) = \{[0 : 1 : 0], [1 : 0 : 0]\}$ which is totally invariant by f_∞ or g_∞ (I is here the exceptional set of f_∞ and g_∞). Let us focus on the first case, as both cases can be understood through the same method.

As in the previous example, we can decompose the Green current in a neighboorhood of L_∞ into T_0 and T_1 both being vertical in some bidisks Δ_0 near $I_0 = [0 : 1 : 0]$ and Δ_1 near $I_1 = [1 : 0 : 0]$. For $\alpha \in \{0,1\}^{\mathbb{N}}$, define $\mathcal{K}_\alpha := \{p \in \mathbb{C}^2,\ \text{s.t}\ \forall k \in \mathbb{N},\ f^k(p) \in \Delta_{\alpha_k}\} \cup \{I_{\alpha(0)}\}$, which is non-empty since the image of any neighborhood of I_i contains L_∞. Let $\mathcal{K}$ be the union of all those sets. Define the matrix $A = (a_{ij})$ by:

$$A = \begin{pmatrix} \frac{n_1+n}{n_1+n_2+n} & \frac{n_2}{n_1+n_2+n} \\ \frac{n_1}{n_1+n_2+n} & \frac{n_2+n}{n_1+n_2+n} \end{pmatrix}$$

We let $\lambda_0 = \frac{n_1}{n_1+n_2}$ and $\lambda_1 = \frac{n_2}{n_1+n_2}$. Then, we define the Borel measure ν on $\{0,1\}^{\mathbb{N}}$ by:

$$\nu(\{\alpha \in \{0,1\}^{\mathbb{N}},\ \alpha(0) = \alpha_0, \dots, \alpha(n) = \alpha_n\}) = \lambda_{\alpha_0} \times \prod_{i=0}^{n-1} a_{\alpha_i \alpha_{i+1}}.$$

We know that ν is invariant and mixing for the left shift σ which defines a subshift on $\{0,1\}^{\mathbb{N}}$ (see Section 3.2). As in Theorem 3.5, we pull back T_0 and T_1 using the horizontal-like maps defined by restricting f to the polydisks (in fact, we must take a large Δ'_i containing Δ_i and the unit circle $|u| = 1$ and use the method of

Section 3.5). Iterating the process, we obtain the following decomposition:

There exists an at most countable set $\Theta \subset \Sigma$ such that for all α in $\Sigma \backslash \Theta$, there exists a current T_α supported on $\overline{\mathcal{K}}_\alpha$ of slice mass 1 such that the Green current T of f admits the following decomposition:

$$T = \int_\Sigma T_\alpha d\nu(\alpha)$$

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